LINEAR DIFFERENCE EQUATIONS AND EXPONENTIAL POLYNOMIALS

BY

WALTER STRODT[1]

Part I. Introduction. The principal result of this paper is Theorem 2, which is an existence and uniqueness theorem for analytic solutions of the difference equation

$$(1) \qquad \sum_{j=1}^{n} A_j y(x + \omega_j) = \phi(x).$$

Important in the proof of Theorem 2, as well as of some interest in itself, is Theorem 1, which states the possibility of securing a particularly simple Mittag-Leffler expansion for the reciprocal of the exponential polynomial

$$(2) \qquad \sum_{j=1}^{n} A_j e^{\omega_j z}.$$

In Theorem 2, equation (1) is studied under the assumptions that $\phi(x)$ is analytic in a sector $\mathfrak{S}(\beta)$: $|\arg x| < \beta \leqq \pi/2$, and that $\omega_1 = 0$, while $\omega_2, \cdots, \omega_n$ lie in $\mathfrak{S}(\beta)$. It is assumed further that there is a non-negative number M such that $\phi(x)$ is *of type* (M, β), by which is meant that $\phi(x)$ is analytic both in $\mathfrak{S}(\beta)$ and also at $x = 0$, and that for every pair of positive numbers ϵ, δ there is a positive number $C_0(\epsilon, \delta)$ such that

$$(3) \qquad |\phi(x)| < C_0(\epsilon, \delta) e^{|x|(M+\epsilon)}$$

when $|\arg x| < \beta - \delta$.

Under these conditions the totality of all those solutions of (1) which are of type (M, β) is shown to be a non-empty finite-parameter family of functions. For this family a representation is found in the form of a sum of contour integrals which are constructed from $\phi(x)$ and from the principal parts of the meromorphic function $1/(\sum_{j=1}^{n} A_j e^{\omega_j z})$. This representation is given by equations (41)–(45) below.

Equation (1) has been studied for real ω_j and real x by Bochner [1][2] and by Raclis [10]. For complex values of ω_j and x it has been studied by Halphén [6] in the case where $\phi(x)$ is entire and of sufficiently small exponential type, by Pincherle [9] in the case where $\phi(x)$ is entire and of arbitrary exponential type, and by Carmichael [3] and by Ghermanesco [5]

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(1) National Research Fellow, at Harvard University and at Columbia University.

(2) Numbers in brackets refer to the bibliography at the end of the paper.

in the case where $\phi(x)$ is any entire function. In addition, Pincherle (loc. cit.) has considered the case where $\phi(x)$ is analytic at infinity, and Ghermanesco (loc. cit.) has considered the case where $\phi(x)$ is meromorphic. *Local* solutions of (1) have been found by Sheffer [11, 12].

In his classic work on the principal solution in the complex plane Nörlund [8, chap. IV] has studied two special cases of equation (1). These are the equations

$$y(x+\omega) - y(x) = \omega\phi(x), \tag{4}$$

and

$$y(x+\omega) + y(x) = 2\phi(x). \tag{5}$$

Nörlund considers these equations under the assumption that either $\phi(x)$ is an entire function of sufficiently small exponential type, or else $\phi(x)$ is analytic in a sector and satisfies a certain growth condition—this growth condition is roughly condition (3) of the present paper, together with a further restriction which consists in requiring that the M of condition (3) be sufficiently small. (In the present paper the number M is unrestricted.)

The basic method of attack which will be employed for Theorem 2 is the use of generalized power-series solutions for certain approximating q-difference equations. This method is a modification of a method introduced by the author [13, 14] in earlier papers.

Part II. Notation. The following notations will be used throughout this paper:

(6) $\omega_1, \cdots, \omega_n$ are given distinct complex numbers.

(7) $A_1, \cdots, A_n$ are given complex numbers, each different from zero.

(8) $f(z)$ is the exponential polynomial $\sum_{j=1}^{n} A_j e^{\omega_j z}$.

(9) $F(z) = 1/(f(z))$.

(10) $Z = \{\zeta_1, \zeta_2, \cdots\}$ is the set of (distinct) zeros of $f(z)$.

(11) j_s is the order of multiplicity of ζ_s as a zero of $f(z)$.

(12) $P_s(z)$, or $\sum_{j=1}^{j_s} B_{sjs}(z-\zeta_s)^{-j}$, is the principal part of $F(z)$ at $z=\zeta_s$ $(s=1, 2, \cdots)$.

(13) For every positive number δ, $Z(\delta)$ is the set of all points whose distance from Z is less than δ.

(14) $\mathcal{G}_1(\delta), \mathcal{G}_2(\delta), \cdots$ are the components of $Z(\delta)$.

(15) $\mathcal{J}_m(\delta)$ is the complete boundary, described in the positive sense, of $\mathcal{G}_m(\delta)$ $(m=1, 2, \cdots)$.

(16) $\bar{\omega}_j$ is the complex conjugate of ω_j.

(17) $\mathcal{P}$ is the closed convex of the points $\bar{\omega}_1, \cdots, \bar{\omega}_n$. (That is, $\mathcal{P}$ is the intersection of all closed half-planes containing $\bar{\omega}_1, \cdots, \bar{\omega}_n$.)

(18) $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_r$ are the edges of $\mathcal{P}$, considered as a closed polygon.

(19) n_j is the number of points of the set $\bar{\omega}_1, \cdots, \bar{\omega}_n$ lying on $\mathcal{E}_j$, $(j=1, 2, \cdots, r)$.

(20) N is the maximum of the numbers n_j $(j=1, 2, \cdots, r)$.

(21) l_j is the length of $\mathcal{E}_j$.

(22) l is the perimeter of $\mathcal{P}$.

(23) $\mathcal{N}_j$ is the half-line through the origin in the direction of the outer normal to $\mathcal{E}_j$ $(j=1, 2, \cdots, r)$.

(24) α_j is the inclination of $\mathcal{N}_j$, with $-\pi<\alpha_j\leqq\pi$ $(j=1, 2, \cdots, r)$.

(25) $\mathcal{S}_0(\beta)$ is the union of $\mathcal{S}(\beta)$ (see introduction) with the set consisting of just one point, the origin.

(26) $\mathcal{N}(M, \beta)$ is defined, when M is non-negative, to be the set of all complex numbers x such that for every number β_1 in the closed interval $(-\beta, \beta)$ the inequality $\Re(x)\cos\beta_1+\Im(x)\sin\beta_1\leqq M$ is valid(3).

(27) b is a positive number greater than unity.

(28) $\mathcal{S}(b, \beta)$ is the set of points x such that either $x=b$, or else for some z in $\mathcal{S}(\beta)$ the equation $x=b(1-e^{-z})$ is valid.

Part III. Exponential polynomials.

A. *The distribution of the zeros.*

LEMMA 1. (a) *There exists a positive number L such that if $\mathcal{H}_j$ is the half-strip*(4) $\{z;\ \Re[z\exp(-i\alpha_j)]\geqq 0,\ |\Im[z\exp(-i\alpha_j)]|<L\}$ $(j=1, 2, \cdots, r)$, *then all zeros of $f(z)$ lie in the union of the $\mathcal{H}_j$. Moreover, for every pair of numbers R_1, R_2 with $0\leqq R_1\leqq R_2$ the number*(5) $N_j(R_1, R_2)$ *of zeros z which lie in $\mathcal{H}_j$ and have $R_1\leqq\Re[z\exp(-i\alpha_j)]\leqq R_2$ satisfies, when R_1 is sufficiently large, the relation*

$$(29)\qquad \begin{aligned}(2\pi)^{-1}l_j(R_2-R_1)-(n_j-1)&\leqq N_j(R_1, R_2)\\ &\leqq(2\pi)^{-1}l_j(R_2-R_1)+(n_j-1).\end{aligned}$$

(b) *If the origin belongs to $\mathcal{P}$, then $f(z)$ is bounded away from zero if z is bounded away from the zeros of $f(z)$. That is, for every positive δ there is a positive $\lambda(\delta)$ such that $|f(z)|\geqq\lambda(\delta)$ whenever z is a point whose distance to the nearest zero of $f(z)$ is at least δ.*

Proof. This lemma follows readily from well known results and methods due to C. E. Wilder, J. D. Tamarkin, and G. Pólya(6).

COROLLARY. *If δ is sufficiently small, $\mathcal{G}_m(\delta)$ contains at most $N-1$ distinct zeros of $f(z)$ $(m=1, 2, \cdots)$.*

Proof. Let $\delta<\pi/(lN)$. By equation (29) there is at most a finite set of values of m such that $\mathcal{G}_m(\delta)$ contains more than $N-1$ zeros of $f(z)$. A sufficient reduction in the size of δ will now bring the desired result.

(3) $\Re$, $\Im$ denote real and imaginary part, respectively.

(4) Evidently $\mathcal{H}_j$ has $\mathcal{N}_j$ for axis of symmetry.

(5) Here each zero is counted a number of times equal to its multiplicity.

(6) The address of R. E. Langer [7] contains an outline of these results and methods, and a bibliography of the theory.

B. *The resolution into partial fractions of the reciprocal of an exponential polynomial.*

THEOREM 1. *Let $\mathfrak{P}$ contain the origin. Let B be any point not in $\mathfrak{Z}$. Then if δ is a sufficiently small positive number, all the following relations are valid:*

(a) $$F(z) = F(B) - \sum_{m=1}^{\infty} \frac{1}{2\pi i} \int_{\mathcal{J}_m(\delta)} \frac{F(T)(z - B)}{(T - z)(T - B)} dT,$$

for every z at positive distance from $\mathfrak{Z}(\delta)$, and

(b) $$F(z) = F(B) + \sum_{m=1}^{\infty} Q_m(z),$$

where

(30) $$Q_m(z) = \sum_{\zeta_s \in \mathcal{G}_m(\delta)} (P_s(z) - P_s(B)),$$

for every z in the complement of $\mathfrak{Z}$, and

(c) *There are at most $N-1$ distinct points ζ_s in $\mathcal{G}_m(\delta)$ $(m=1, 2, \cdots)$, and*

(d) *The length of $\mathcal{J}_m(\delta)$ is not more than $2\pi(N-1)\delta$, and*

(e) *There exists a positive number $\lambda(\delta)$ such that $|F(T)| < 1/\lambda(\delta)$ for all T on $\mathcal{J}_m(\delta)$ $(m=1, 2, \cdots)$, and*

(f) *The series in* (a) *converges absolutely, and the corresponding series of absolute values converges uniformly, for z in any bounded set at positive distance from $\mathfrak{Z}(\delta)$, and the series in* (b) *converges absolutely, and the corresponding series of absolute values converges uniformly, for z in any bounded set at positive distance from $\mathfrak{Z}$.*

Proof. (c) is an immediate consequence of the corollary to Lemma 1. (e) follows immediately from Lemma 1b. (d) follows immediately from (c). If δ is less than the distance from B to $\mathfrak{Z}$, then (b) is obviously implied by (a). It remains, then, to establish (a) and (f). We shall use Cauchy's method of resolving a meromorphic function into partial fractions.

Let z be any point not in $\mathfrak{Z}$.

Now (e) is valid for every δ, and we assume δ sufficiently small so that (c) and (d) are true, and sufficiently small so that z and B are at positive distance from $\mathfrak{Z}(\delta)$.

Let R be a positive number, so large that $|z| < R$ and $|B| < R$ and $\delta < R$. Let $\mathcal{U}_R$ be the union of the circle $|z| < R$ with all the sets $\mathcal{G}_m(\delta)$ which have points in common with that circle. Let $\mathcal{K}_R$ be the complete boundary of $\mathcal{U}_R$ described in the positive sense.

Let $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_k$ be the components of $\mathfrak{Z}(\delta)$ which are included in $\mathcal{U}_R$ but not wholly included in the circle $|z| < R$. Then there is a zero ξ_j of $f(z)$ in $\mathcal{T}_j$ $(j=1, 2, \cdots, k)$, which lies in the ring $R-\delta < |z| < R+\delta$.

We assume now that δ is less than πl^{-1} (see Notations (20), (22)). Then it follows easily from (29), with $R_2 = R+\delta$ and with $R_1 = [(R-\delta)^2 - L^2]^{1/2}$ (L as in Lemma 1a), that if R is sufficiently large, the number of zeros in the ring $R-\delta < |z| < R+\delta$ is less than $(2\pi)^{-1} l 2\pi (l)^{-1} + N_1$ where $N_1 = \sum_{j=1}^{r} (n_j - 1)$, and therefore $k \leqq N_1$.

Since each $\mathcal{T}_j$ is the union of at most $N-1$ circles of radius δ, the length of $\mathcal{K}_R$ is at most $2\pi R + 2\pi(N-1)\delta N_1$, which is less than $7R$ if R is large.

Hence

$$\left| \frac{1}{2\pi i} \int_{\mathcal{K}_R} \frac{F(T)(z-B)}{(T-z)(T-B)} dT \right| \leqq \frac{1}{2\pi}(7R) \frac{1}{\lambda(\delta)} \frac{|z| + |B|}{(R-|z|)(R-|B|)}, \tag{31}$$

which approaches zero as R becomes infinite.

But

$$\frac{1}{2\pi i} \int_{\mathcal{K}_R} \frac{F(T)(z-B)}{(T-z)(T-B)} dT = F(z) - F(B) + \sum_{\mathcal{G}_{m(\delta)} \subset \mathcal{V}_R} \frac{1}{2\pi i} \int_{\mathcal{J}_{m(\delta)}} \frac{F(T)(z-B)}{(T-z)(T-B)} dT. \tag{32}$$

Hence (a) is valid if the series in (a) converges. But (29) implies that $(T-z)^{-1}(T-B)^{-1} = O(m^{-2})$ on $\mathcal{J}_m(\delta)$, and this, together with (d) and (e), implies that the series in (a) converges, and implies statement (f).

REMARK. Theorem 1 may, in part, be summarized in the statement that if B is any point at which $F(z)$ is finite, then the series $\sum_{s=1}^{\infty} (P_s(z) - P_s(B))$, *provided the terms are properly bracketed*, converges to $F(z) - F(B)$. In the simple case of commensurate real exponents ω_j it is plain that if the origin belongs to $\mathcal{P}$, then no bracketing is needed. In the general case where the origin belongs to $\mathcal{P}$ this bracketing is essential. Indeed, to secure convergence of the Mittag-Leffler expansion $\sum_{s=1}^{\infty} (P_s(z) - P_s^*(z))$, where $P_s^*(z)$ is the sum of the first k_s terms in the Taylor's series expansion of $P_s(z)$ at $z = B$, it is in general impossible to choose values for k_s which remain bounded as s becomes infinite. This, together with stronger statements on the nonconvergence of the unbracketed series $\sum_{s=1}^{\infty} (P_s(z) - P_s(B))$, has been shown by Borel [2] for the particular function $F(z) = 1/(f(z))$ with $f(z)$ the exponential polynomial $\sin \pi z \sin \alpha\pi z$, α being a carefully chosen irrational real number.

Borel remarks upon the usefulness, in the particular case cited, of bracketing the principal parts. Whittaker [15] has made systematic use of such bracketing in the study of meromorphic functions of finite order. However, Whittaker uses a plan of bracketing different from the one used in the present paper, and his results do not specialize in the case of the reciprocal of an exponential polynomial to Theorem 1 of the present paper.

Part IV. Further preliminary theorems.

Lemma 2. *Let* $\omega_1=0$ *and let* $\omega_2, \cdots, \omega_n$ *be distinct points of* $\mathfrak{S}(\beta)$. *Let* $\mathfrak{P}$, $\alpha_1, \cdots, \alpha_r$ *be defined by* (17) *and* (24). *Then* $-(\beta+\pi/2)<\alpha_s<(\beta+\pi/2)$ $(s=1, 2, \cdots, r)$.

Proof. This is geometrically apparent. We omit the analytic proof, which is easily supplied.

Lemma 3. *If* $M_1<M_2$, *and if* z_1 *is in* $\mathfrak{N}(M_1, \beta)$, *while* z_2 *is in the complement of* $\mathfrak{N}(M_2, \beta)$, *then* $|z_2-z_1|>M_2-M_1$.

Proof. This is obvious.

Lemma 4. *Let* $\omega_1=0$, *and let* $\omega_2, \cdots, \omega_n$ *be distinct points of* $\mathfrak{S}(\beta)$. *Then*:

(a) *If* M *is any non-negative number there are at most finitely many zeros of* $f(z)$ *in* $\mathfrak{N}(M, \beta)$.

(b) *If* $\zeta_1, \cdots, \zeta_p$ *are the zeros of* $f(z)$ *in* $\mathfrak{N}(M, \beta)$, *then there is a real number* M_1 *greater than* M *such that* $\zeta_1, \cdots, \zeta_p$ *are the only zeros of* $f(z)$ *in* $\mathfrak{N}(M_1, \beta)$.

(c) *If* $\zeta_1, \cdots, \zeta_p$ *are the zeros of* $f(z)$ *in* $\mathfrak{N}(M, \beta)$, *and* $\zeta_{p+1}, \zeta_{p+2}, \cdots$ *are the other zeros of* $f(z)$, *then there is a positive distance from the set* $\mathfrak{N}(M, \beta)$ *to the set* $\{\zeta_{p+1}, \zeta_{p+2}, \cdots\}$.

(d) *Let* $\zeta_1, \cdots, \zeta_p$ *be the zeros of* $f(z)$ *in* $\mathfrak{N}(M_1, \beta)$. *Let* $\zeta_{p+1}, \zeta_{p+2}, \cdots$ *be numbered in any order of nondecreasing modulus. Let* δ_0 *be a positive number sufficiently small so that if* $\delta<\delta_0$, *then the component of* $\mathfrak{Z}(\delta)$ *which contains* ζ_s *contains no other point of* $\mathfrak{Z}$ $(s=1, 2, \cdots, p)$, *and* $\mathcal{G}_m(\delta)$ *contains at most* $N-1$ *distinct points of* $\mathfrak{Z}$ $(m=1, 2, \cdots)$. *For every* δ *less than* δ_0 *let the components* $\mathcal{G}_m(\delta)$ *of* $\mathfrak{Z}(\delta)$ *be numbered in such a way that* $\mathcal{G}_m(\delta)$ *contains a* ζ_j *for which* $j\geqq m$. (*For example, let* $\mathcal{G}_1(\delta)$ *be chosen to contain* ζ_1, *and after* $\mathcal{G}_s(\delta)$ $(s\leqq k)$ *have been chosen, let* $\mathcal{G}_{k+1}(\delta)$ *be chosen to contain the* ζ_j *of minimum subscript lying in the complement of* $\mathcal{G}_1(\delta)+\cdots+\mathcal{G}_k(\delta)$.)

Then there is a positive number D, *and a finite set of real numbers* $\Gamma_1, \cdots, \Gamma_h$, *lying in the open interval* $(-\beta, \beta)$, *such that for every sufficiently small positive* δ *a sequence of numbers* $\gamma_{p+1}, \gamma_{p+2}, \cdots$ *can be chosen, with every* γ_m *in the set* $(\Gamma_1, \cdots, \Gamma_h)$, *and with the inequality*

$$\Re(T)\cos\gamma_m-\Im(T)\sin\gamma_m>M_1+Dm \tag{33}$$

valid if T *is on* $\mathcal{J}_m(\delta)$ *and* $m>p$.

(e) *If* $\zeta_1, \cdots, \zeta_p$ *are the zeros of* $f(z)$ *in* $\mathfrak{N}(M_1, \beta)$, *then there is a positive number* β_0 *smaller than* β *such that* $\zeta_1, \cdots, \zeta_p$ *are the zeros of* $f(z)$ *in* $\mathfrak{N}(M_1, \beta_0)$.

Proof. (a) By Lemma 2 the outer normals to the sides of $\mathfrak{P}$ have their inclinations in the open interval $(-[\beta+\pi/2], [\beta+\pi/2])$. Let β_2 be chosen smaller than β and such that the inclinations of the outer normals to $\mathfrak{P}$ lie in the open interval $(-[\beta_2+\pi/2], [\beta_2+\pi/2])$. Then by Lemma 1 all but finitely many of the zeros of $f(z)$ lie in $\mathfrak{S}(\beta_2+\pi/2)$. Since the intersection of $\mathfrak{S}(\beta_2+\pi/2)$ and $\mathfrak{N}(M, \beta)$ is clearly a bounded set, part (a) follows.

(b) By application of part (a) with M replaced by M_2, we obtain the result that $\mathfrak{N}(M_2, \beta)$ will, for any M_2, contain only finitely many zeros of $f(z)$. Let M_2 be any number greater than M. Evidently the zeros of $f(z)$ in $\mathfrak{N}(M_2, \beta)$ can be denoted by $\zeta_1, \zeta_2, \cdots, \zeta_t$, with $t \geqq p$. For every k with $p<k \leqq t$ we have a β_k in the closed interval $(-\beta, \beta)$ such that $\Re(\zeta_k) \cos \beta_k + \Im(\zeta_k) \cdot \sin \beta_k > M$. Choosing M_1 as any number which is less than the minimum of the numbers $\Re(\zeta_k) \cos \beta_k + \Im(\zeta_k) \sin \beta_k$ $(k=p+1, \cdots, t)$, and which is greater than M, we see that all the points $\zeta_{p+1}, \cdots, \zeta_t$ are in the complement of $\mathfrak{N}(M_1, \beta)$. This establishes (b).

(c) This is an immediately consequence of (b) and Lemma 3.

(d) In view of Lemma 1, there exists a positive number K, a function $s(m)$ defined for $m=1, 2, \cdots$ and assuming values in the set $(1, \cdots, r)$, a sequence of non-negative numbers $\rho_1, \rho_2, \cdots$, and a sequence of complex numbers $\theta_1, \theta_2, \cdots$ satisfying $|\theta_m| < K$ $(m=1, 2, \cdots)$, such that

$$\zeta_m = \rho_m \exp (i\alpha_{s(m)}) + \theta_m. \tag{34}$$

Now because of Lemma 2 there exists for every s in the set $(1, \cdots, r)$ a number β_s, in the open interval $(-\beta, \beta)$, such that $\cos(\alpha_s+\beta_s)>0$. If $s=s(m)$,

$$\begin{aligned} \Re[\zeta_m \exp (i\beta_s)] &= \Re\{[\exp (i\beta_s)][\rho_m \exp (i\alpha_s) + \theta_m]\} \\ &\geqq \rho_m \cos (\alpha_s + \beta_s) - K. \end{aligned} \tag{35}$$

Let M_2 be a positive number larger than M_1, such that $\zeta_1, \cdots, \zeta_p$ are the zeros of $f(z)$ in $\mathfrak{N}(M_2, \beta)$. Evidently, from (35), there is a positive integer m_0 such that $\Re(\zeta_m \exp (i\beta_s)) > M_2$ if $m > m_0$, and $s=s(m)$. We define g_m to be $\beta_{s(m)}$, if $m>m_0$. Then if $m>m_0$ we have

$$\Re(\zeta_m) \cos g_m - \Im(\zeta_m) \sin g_m > M_2. \tag{36}$$

We consider next the integer m in the set $p+1, \cdots, m_0$. For each such m let g_m be a number in the open interval $(-\beta, \beta)$ such that (36) holds $(m=p+1, \cdots, m_0)$. The existence of such g_m follows from the fact that the points $\zeta_{p+1}, \cdots, \zeta_{m_0}$ are in the complement of $\mathfrak{N}(M_2, \beta)$.

Now it follows from Lemma 1 that there is a positive number D' such that $\rho_m > D'm$ for all sufficiently large m. Hence from (35) we conclude that there is a positive number D'' such that

$$\Re(\zeta_m) \cos g_m - \Im(\zeta_m) \sin g_m > D''m, \tag{37}$$

for all sufficiently large m. From (37) and (36) it follows that there is a positive D such that

(38) $$\Re(\zeta_m)\cos g_m - \Im(\zeta_m)\sin g_m > M_2 + Dm$$

if $m>p$.

Let δ_1 be a positive number not exceeding δ_0 and such that $2(N-1)\delta_1$ is less than M_2-M_1. Let δ be a positive number less than δ_1. We define the sequence $\gamma_{p+1}, \gamma_{p+2}, \cdots$ as follows: For each integer m greater than p we choose an integer j not less than m such that ζ_j is a point of $\mathcal{Z}$ which lies in $\mathcal{G}_m(\delta)$, and we define $\gamma_m=g_j$. Then if T is on $\mathcal{J}_m(\delta)$, $|T-\zeta_j|\leqq(2(N-1)\delta<M_2-M_1$. Hence $|[\Re(T)-\Re(\zeta_j)]\cos g_j-[\Im(T)-\Im(\zeta_j)]\sin g_j|<M_2-M_1$. But $\Re(T)\cos\gamma_m-\Im(T)\sin\gamma_m=\Re(\zeta_j)\cos g_j-\Im(\zeta_j)\sin g_j+[\Re(T)-\Re(\zeta_j)]\cdot\cos g_j-[\Im(T)-\Im(\zeta_j)]\sin g_j\geqq M_2+Dj-(M_2-M_1)=M_1+Dj\geqq M_1+Dm$.

This proves (d), with $\{\Gamma_1, \cdots, \Gamma_h\}=\{\beta_1, \cdots, \beta_r; g_{p+1}, \cdots, g_{m_0}\}$.

(e) Since $\mathcal{N}(M_1, \beta_0)\supset\mathcal{N}(M_1, \beta)$, it suffices to prove that there is a positive β_0 less than β such that ζ_j is in the complement of $\mathcal{N}(M_1, \beta_0)$, whenever $j>p$. Recalling that the set $(\gamma_{p+1}, \gamma_{p+2}, \cdots)$ is a finite set, let $\beta_0=\max|\gamma_m|$ $(m=p+1, p+2, \cdots)$. Then $\beta_0<\beta$. Now it follows from (38), which holds for $m=p+1, p+2, \cdots$, that ζ_j $(j=p+1, p+2, \cdots)$ is in the complement of $\mathcal{N}(M_1, \beta_0)$.

LEMMA 5. *Let η, X be any positive numbers. Then $X\sin\eta<e^{\eta X}-\cos\eta$.*

Proof. This is obvious.

LEMMA 6. *Let $0<\beta\leqq\pi/2$. Let $z\in\mathcal{S}(\beta)$. Then $1-e^{-z}\in\mathcal{S}(\beta)$.*

Proof. Let $\zeta=1-e^{-z}$. Let $z=\rho(\cos\beta_1\pm i\sin\beta_1)$, with $0\leqq\beta_1<\beta$, and $\rho>0$. If $\beta_1=0$, $\zeta>0$, and hence $\zeta\in\mathcal{S}(\beta)$. If $\beta_1>0$, we define η by the equation $\eta=\rho\sin\beta_1$. Then $z=\eta\cot\beta_1\pm i\eta$. Hence $\zeta=(1-e^{-\eta\cot\beta_1}\cos\eta)\pm ie^{-\eta\cot\beta_1}\sin\eta$. Hence $|\Im(\zeta)/\Re(\zeta)|\cot\beta_1=(\cot\beta_1)\sin\eta(e^{\eta\cot\beta_1}-\cos\eta)^{-1}$ and by Lemma 5, with $X=\cot\beta_1$, the last number is less than unity, so that $|\Im(\zeta)/\Re(\zeta)|<\tan\beta_1$. This relation, together with the obvious relation $\Re(\zeta)>0$, implies $\zeta\in\mathcal{S}(\beta_1)\subset\mathcal{S}(\beta)$.

LEMMA 7. (a) *For every b, $\mathcal{S}(b, \beta)$ is a bounded set.*

(b) *For every b, $\mathcal{S}(b, \beta)\subset\mathcal{S}(\beta)$.*

(c) *If $\mathfrak{F}$ is a closed bounded subset of $\mathcal{S}(\beta)$, then when b is sufficiently large $\mathfrak{F}\subset\mathcal{S}(b, \beta)$.*

(d) *$\mathcal{S}(b, \beta)$ contains a neighborhood of b.*

(e) *If $x\in\mathcal{S}(b, \beta)$ and if $z\in\mathcal{S}(\beta)$, then $b+e^{-z}(x-b)\in\mathcal{S}(b, \beta)$.*

Proof. (a) It is evident that if $x\in\mathcal{S}(b, \beta)$, then $|x|<2b$.

(b) This follows immediately from Lemma 6.

(c) Let $\mathfrak{F}$ be any closed bounded subset of $\mathcal{S}(\beta)$. Let X be a positive number such that $|x|<X$ whenever $x\in\mathfrak{F}$. Let $x\in\mathfrak{F}$. Let $b>1$, let $q=1-b^{-1}$, and define z by the equation $b(1-q^z)=x$. Then(7)

(7) For definiteness, throughout this paper, $\log(1-b^{-1}x)$ is understood to be that single-valued branch in the set $\arg(1-b^{-1}x)\neq\pi$ which is real when $\arg(1-b^{-1}x)=0$, and $\log q$ is understood to be a negative number.

(39) $$z = \frac{\log(1 - x/b)}{\log q}.$$

Thus, since $d[\log(1-xb^{-1})-x\log q]/dq = x(x-1)(1-q)(1-x+xq)^{-1}q^{-1}$, we have

(40) $$z - x = \left(\int_1^q \frac{x(x-1)(1-t)}{(1-x+xt)t}\,dt\right)\Big/\left(\int_1^q t^{-1}dt\right),$$

whence, if $b > X_1 > X$, then $|z-x| < X(X+1)/(X_1-X)$. (For $|(1-t)/(1-x+xt)| < (1/b)/[1-(X/b)] < 1/(X_1-X)$.) Thus for every positive ϵ there is a positive X_1 such that if x is in $\mathfrak{F}$ and z is defined by (39), and if $b > X_1$, then $|z-x| < \epsilon$. If ϵ is less than the distance from $\mathfrak{F}$ to the boundary of $\mathfrak{S}(\beta)$, then z is in $\mathfrak{S}(\beta)$. But $b(1-q^z) = b\{1-\exp[-z(-\log q)]\}$, and if z is in $\mathfrak{S}(\beta)$, so is $z(-\log q)$. Hence $x \in \mathfrak{S}(b, \beta)$, for all x in $\mathfrak{F}$, if $b > X_1$.

(d) Let $\rho = be^{-\pi \cot \beta}$. Let $\mathcal{G}$ be the set $\{x; |x-b| < \rho\}$. Let $x \in \mathcal{G}$. If $x = b$, then $x \in \mathfrak{S}(b, \beta)$ by definition. If $x \neq b$, let $x = b - Re^{i\theta}$, with $0 < R < \rho$, and $-\pi < \theta \leqq \pi$. Then $x = b(1-e^{-z})$, with $z = \log(b/R) - i\theta$. Since $\Re(z) > \log(b/\rho) = \pi \cot \beta$, and $|\Im(z)| = |\theta| \leqq \pi$, it follows that $z \in \mathfrak{S}(\beta)$. Thus $x \in \mathfrak{S}(b, \beta)$. Hence $\mathcal{G} \subset \mathfrak{S}(b, \beta)$.

(e) Let $x = b(1-e^{-\zeta})$, where $\zeta \in \mathfrak{S}(\beta)$. Then $b + e^{-z}(x-b) = b(1-e^{-(z+\zeta)})$, but since z and ζ are both in $\mathfrak{S}(\beta)$, so is $z+\zeta$.

LEMMA 8. *Let $b > 1$. Let $q = 1-b^{-1}$. Then $|b \log q + 1| < (2qb)^{-1}$ and $1 < b|\log q| < q^{-1}$.*

The proof is obvious.

LEMMA 9. *Let x be any complex number. Let z be in $\mathfrak{S}_0(\beta)$. Let $b > 1$. Let $q = 1-b^{-1}$. Let $\zeta = b + (x-b)\exp(z \log q)$. Then*

(a) $|e^{z \log q} - 1| < |z|/(bq)$,

(b) $|\zeta| \leqq |x| + q^{-1}|z|$,

(c) $|\zeta - (x+z)| \leqq |z|(|x|+|z|+1)/(bq^2)$.

Proof. (a) $|\exp(z\log q) - 1| = |z \log q \int_0^1 \exp(tz \log q)\, dt|$. Hence, since $\Re(tz \log q) \leqq 0$, it follows that $|\exp(z \log q) - 1| \leqq |z|\,|\log q| \leqq |z|/(bq)$.

(b) $\zeta = x \exp(z \log q) + b[1-\exp(z\log q)]$, and therefore, by (a), we have $|\zeta| \leqq |x| + q^{-1}|z|$.

(c) $\zeta - (x+z) = (x-b)\ [\exp(z \log q) - 1] - z = z\int_0^1 [(x-b)(\log q)\exp(tz\log q) - 1]dt = z\int_0^1 \{x(\log q)\exp(tz \log q) - (b \log q + 1) + b \log q \cdot [1-\exp(tz \log q)]\}dt$. Hence $|\zeta - (x+z)| \leqq |z|\{|x|(bq)^{-1} + (2bq)^{-1} + q^{-1}|z|(bq)^{-1}\} \leqq |z|(|x|+|z|+1)/(bq^2)$.

LEMMA 10. *Let x, ξ be any two complex numbers, and let b be a positive number greater than 1, and greater than $|\xi| + |x-\xi|$. Let $q = 1-b^{-1}$. Let $V = [\log(x-b) - \log(\xi - b)]/(\log q)$. Then:*

(a) $|V| \leqq b|x-\xi|(b-|\xi|-|x-\xi|)^{-1}$.
(b) $|V-(x-\xi)| \leqq |x-\xi|(|x-\xi-1|+|\xi|)q^{-1}(b-|\xi|-|x-\xi|)^{-1}$.

Proof. (a) $V=(x-\xi)(\log q)^{-1}(\xi-b)^{-1}\int_0^1[1+(x-\xi)(\xi-b)^{-1}t]^{-1}dt$. Since (by Lemma 8) $|\log q|>b^{-1}$, we obtain

$$\begin{aligned}|V| &\leqq |x-\xi|(1-|\xi|b^{-1})^{-1}[1-|x-\xi|(b-|\xi|)^{-1}]^{-1}\\ &= b|x-\xi|(b-|\xi|-|x-\xi|)^{-1}.\end{aligned}$$

(b) Since $\log q=\log(1-b^{-1})=-\int_0^1(b-t)^{-1}dt$, we have

$$\begin{aligned}V-(x-\xi) &= (x-\xi)(\log q)^{-1}\int_0^1\{[(\xi-b)+(x-\xi)t]^{-1}+[b-t]^{-1}\}dt\\ &= (x-\xi)(\log q)^{-1}\int_0^1\{[(x-\xi-1)t+\xi]\\ &\qquad\cdot[\xi-b+(x-\xi)t]^{-1}[b-t]^{-1}\}dt,\end{aligned}$$

from which (b) follows readily.

Lemma 11. *Let $\zeta_1, \cdots, \zeta_p$ be any given complex numbers. Let $j_1, \cdots, j_p$ be given positive integers. Let b be a real variable greater than* 1. *Let $C_{sj}(b)$ $(s=1, \cdots, p;\ j=0, 1, \cdots, j_s-1)$ be complex-valued functions of b. Let $q=1-b^{-1}$. Let $U=\log(1-b^{-1}x)/(\log q)$.* Let

$$H(x, b) = \sum_{s=1}^{p}\sum_{j=0}^{j_s-1} C_{sj}(b)U^je^{\zeta_sU}.$$

Then if there is an open region $\mathcal{G}$ such that as b becomes infinite $H(x, b)$ approaches a limit function $I(x)$ uniformly in $\mathcal{G}$, $I(x)$ must be of the form

$$I(x) = \sum_{s=1}^{p}\sum_{j=0}^{j_s-1} C_{sj}x^je^{\zeta_sx}$$

for some constants C_{sj}.

Proof. Assume that $\mathcal{G}$ is bounded. (The general case follows, by analytic continuation, from the bounded case.) Now $x=(1-q^U)(1-q)^{-1}$. As b becomes infinite, U approaches x uniformly in $\mathcal{G}$. This follows at once from Lemma 10b, with $\xi=0$. Let D_x, D_U be symbols denoting operations of differentiation with respect to x, U respectively. Let $\mathcal{C}$ be a circle which with its boundary is included in $\mathcal{G}$. Now $(D_U-\zeta_1)^{j_1}\cdots(D_U-\zeta_p)^{j_p}H(x, b)=0$. Hence $(D_U-\zeta_1)^{j_1}\cdots(D_U-\zeta_p)^{j_p}I(x)$ is small throughout $\mathcal{C}$, if b is large, since $H(x, b)$ approaches $I(x)$ uniformly in $\mathcal{G}$ as b becomes infinite. Also $(D_U-\zeta_1)^{j_1}\cdots(D_U-\zeta_p)^{j_p}I(x)-(D_x-\zeta_1)^{j_1}\cdots(D_x-\zeta_p)^{j_p}I(x)$ is small throughout $\mathcal{C}$ if b is large. (For d^sU/dx^s is near d^sx/dx^s, for each s, uniformly in $\mathcal{C}$, if b is large.) Hence $(D_x-\zeta_1)^{j_1}\cdots(D_x-\zeta_p)^{j_p}I(x)$, which is independent of b, must be zero throughout $\mathcal{C}$. Hence $I(x)$ has the asserted form.

LEMMA 12. *If $\phi(x)$ is any function of type (M, β) (as defined in the introduction), then $\phi'(x)$ is of type (M, β).*

Proof. This follows readily from the Cauchy integral formula.

Part V. Linear difference equations.

THEOREM 2. *Given equation (1) with $\omega_1=0$ and with $\omega_2, \cdots, \omega_n$ in $\mathfrak{S}(\beta)$. Given that $\phi(x)$ is of type (M, β) (as defined in the introduction). Let B be any complex number such that $f(B)\neq 0$. Then there exists a solution $y_0(x)$ of (1) which is of type (M, β). Moreover, if $\zeta_1, \cdots, \zeta_p$ are the distinct zeros(8) of $f(z)$ in $\mathfrak{N}(M, \beta)$ with respective orders $j_1, \cdots, j_p$, then every solution $y(x)$ of (1) which is of type (M, β) is of the form*

$$y_0(x) + \sum_{s=1}^{p} \sum_{j=0}^{j_s-1} C_{sj} x^j e^{\zeta_s x} \tag{41}$$

where the C_{sj} are constants, and conversely every function of the form (41) is a solution of (1) of type (M, β). Finally, there exist contours $\mathcal{L}_{p+1}, \mathcal{L}_{p+2}, \cdots$, each being a half-line lying in $\mathfrak{S}_0(\beta)$ with the origin for initial point, and the set of these half-lines being a finite set, such that if δ is any sufficiently small positive number and if the contours $\mathcal{J}_m(\delta)$ $(m=1, 2, \cdots)$ are properly numbered, then

$$y_0(x) = g\phi(x) + \sum_{s=1}^{p} G_s^{**}(x) - \sum_{m=p+1}^{\infty} Y_m(x), \tag{42}$$

where

$$g = F(B) - \sum_{s=1}^{p} P_s(B), \tag{43}$$

and

$$G_s^{**}(x) = \int_0^x (2\pi i)^{-1} \int_{\mathcal{J}_s(\delta)} e^{(x-\xi)T}\phi(\xi)F(T)dTd\xi, \tag{44}$$

$(s=1, 2, \cdots, p)$, and

$$\begin{aligned} Y_m(x) = (2\pi i)^{-1} \int_{\mathcal{J}_m(\delta)} & F(T)(T-B)^{-1} \\ & \cdot \int_{\mathcal{L}_m} e^{-zT}[\phi'(x+z) - B\phi(x+z)]dzdT, \end{aligned} \tag{45}$$

$(m=p+1, p+2, \cdots)$.

The construction of these contours $\mathcal{L}_m$ depends upon M, but is otherwise independent of $\phi(x)$.

(8) There are only finitely many such zeros, by Lemma 4a.

Proof. Let $\zeta_{p+1}, \zeta_{p+2}, \cdots$ be numbered in any order of nondecreasing modulus. Let M_1 be a positive number greater than M and such that $\zeta_{p+1}, \zeta_{p+2}, \cdots$ are all in the complement of $\mathcal{N}(M_1, \beta)$. (The existence of such an M_1 follows from Lemma 4b.) Let δ be a positive number so small that:

(46) the distance from the set $(\zeta_{p+1}, \zeta_{p+2}, \cdots)$ to the set $\mathcal{N}(M_1, \beta)$ is greater than 2δ (cf Lemma 4c),

(47) the minimum distance between distinct points of the set $\zeta_1, \cdots, \zeta_p$ is greater than 2δ,

(48) each component of $\mathcal{Z}(\delta)$ contains at most $N-1$ (distinct) zeros of $f(z)$,

(49) the distance from B to $\mathcal{Z}$ is greater than δ.

Because of (46), (47) the component of $\mathcal{Z}(\delta)$ containing ζ_s $(s=1, 2, \cdots, p)$ contains no other point of $\mathcal{Z}$. Let the components of $\mathcal{Z}(\delta)$ be numbered in any fashion such that $\mathcal{G}_m(\delta)$ contains a ζ_j for which $j \geqq m$. (Cf. Lemma 4d.) Then evidently the component of $\mathcal{Z}(\delta)$ which contains ζ_s is $\mathcal{G}_s(\delta)$ $(s=1, 2, \cdots, p)$.

Let b be a positive number greater than 1. Let $q=1-b^{-1}$. Let $T_1x, T_2x, \cdots, T_nx$ be functions of x defined as follows:

$$T_jx = q^{\omega_j}(x-b)+b \qquad (j=1, 2, \cdots, n), \tag{50}$$

where by q^{ω_j} is meant $e^{\omega_j \log q}$. We consider the functional equation

$$\sum_{j=1}^{n} A_j y(T_jx) = \phi(x), \tag{51}$$

which, for large b, is in a formal sense an approximation to equation (1), since $q^{\omega_j}(x-b)+b$ is, for fixed x, near $x+\omega_j$ if b is large. (Cf. Lemma 9c with $z=\omega_j$.)

One may verify without difficulty that if $f(z)$ has a zero of order i_k at $z=k \log q$ $(k=0, 1, \cdots)$(9), and if the Taylor's series expansion of $\phi(x)$ at $x=b$ is $\sum_{k=0}^{\infty} \phi_k(x-b)^k$, then the function $y(x, b)$ defined by

$$\begin{aligned} y(x, b) = {} & \sum_{k=0}^{\infty} \frac{\phi_k(x-b)^k}{f^{(i_k)}(k \log q)} \left(\frac{\log(1-b^{-1}x)}{\log q} \right)^{i_k} \\ & + \sum_{s=1}^{p} \sum_{j=0}^{i_s-1} C_{sj} \left(\frac{\log(1-b^{-1}x)}{\log q} \right)^{j} (1-b^{-1}x)^{\zeta_s/\log q}, \end{aligned} \tag{52}$$

where the C_{sj} are arbitrary constants, is a solution of (51)(10). Let σ_s be defined as follows:

$$\sigma_s = \zeta_s/(\log q) \qquad (s=1, \cdots, p). \tag{53}$$

Let ψ_s be defined as follows:

(9) Only finitely many i_k can be different from zero, since, by Lemma 4a, $f(z)$ has at most finitely many negative zeros.

(10) Cf. the footnote to equation (39). By $(1-b^{-1}x)^{\zeta_s/\log q}$ is meant $\exp\left[\zeta_s(\log q)^{-1} \cdot \log(1-b^{-1}x)\right]$.

(54) $\psi_s=\phi_{\sigma_s}$ if σ_s is a non-negative integer; $\psi_s=0$ otherwise. Then

$$(55)\quad \begin{aligned} y(x, b) = &\sum_{k\notin(\sigma_1,\cdots,\sigma_p)} \phi_k(x-b)^k F(k\log q) \\ &+\sum_{s=1}^{p}\frac{\psi_s(x-b)^{\sigma_s}}{f^{(j_s)}(\zeta_s)}\left(\frac{\log(1-b^{-1}x)}{\log q}\right)^{j_s} + H(x, b) \end{aligned}$$

where[11]

$$(56)\quad H(x, b) = \sum_{s=1}^{p}\sum_{j=0}^{j_s-1} C_{sj}\left(\frac{\log(1-b^{-1}x)}{\log q}\right)^{j}(1-b^{-1}x)^{\sigma_s}.$$

Now if k is not in the set $(\sigma_1, \cdots, \sigma_p)$ then

$$(57)\quad F(k\log q) = F(B) + \sum_{m=1}^{p} Q_m(k\log q) + \sum_{m=p+1}^{\infty} Q_m(k\log q)$$

by Theorem 1b. Hence, if k is not in the set $(\sigma_1, \cdots, \sigma_p)$, then

$$(58)\quad \begin{aligned} F(k\log q) = &F(B) + \sum_{s=1}^{p}(P_s(k\log q) - P_s(B)) \\ &- \sum_{m=p+1}^{\infty}(2\pi i)^{-1}\int_{\mathcal{J}_m(\delta)} \frac{F(T)(k\log q - B)}{(T-k\log q)(T-B)}\,dT. \end{aligned}$$

Thus

$$(59)\quad y(x, b) = g\phi(x) + \sum_{s=1}^{p} G_s(x, b) - \sum_{m=p+1}^{\infty} Y_m(x, b) + H(x, b)^{(12)},$$

where

$$(60)\quad g = F(B) - \sum_{s=1}^{p} P_s(B),$$

and

$$(61)\quad \begin{aligned} G_s(x, b) = &\sum_{k\notin(\sigma_1,\cdots,\sigma_p)} \phi_k(x-b)^k P_s(k\log q) \\ &+\frac{\psi_s(x-b)^{\sigma_s}}{f^{(j_s)}(\zeta_s)}\left(\frac{\log(1-b^{-1}x)}{\log q}\right)^{j_s} \end{aligned}$$

$(s=1, 2, \cdots, p)$, and

(11) In the sequel the symbol $H(x, b)$ will be used *generically* to denote functions which are of the form (56) for some constants C_{sj}, arbitrary or otherwise, in accordance with the text. (Each C_{sj} may vary with b.)

(12) This $H(x, b)$ may be different from that in (55). The change in $H(x, b)$ will take care of certain changes which occur in the range of the subscripts k, in the passage from (55) to (59)–(62).

$$(62)\quad Y_m(x, b) = (2\pi i)^{-1} \int_{\mathcal{J}_m(\delta)} \frac{F(T)}{T - B} \sum_{k=0}^{\infty} \left(\frac{k \log q - B}{T - k \log q} \right) \phi_k (x - b)^k dT.$$

(The order of limiting operations may be changed, as done just above, because of the convergence of the series $\sum_{k=0}^{\infty} |\phi_k| \, |x-b|^k \sum_{m=p+1}^{\infty} J_m(k, T)$, where $J_m(k,T) = (2\pi)^{-1} \int_{\mathcal{J}_m(\delta)} |F(T)| \, |k \log q - B| \, |T-b|^{-1} |T-k \log q|^{-1} |dT|$. In this connection we note that $|F(T)| < 1/\lambda(\delta)$ on $\mathcal{J}_m(\delta)$ (by Lemma 1b), that if the $\mathcal{J}_m(\delta)$ are properly numbered then on $\mathcal{J}_m(\delta) |T-B|^{-1} = O(m^{-1})$ and $|T-k \log q| = O(m^{-1})$, because of Lemma 1a, and that the length of $\mathcal{J}_m(\delta)$ is bounded $(m = p+1, p+2, \cdots)$, because of (48).

We seek next to write $(T-k \log q)^{-1}$ in a more convenient form, using the equation

$$(63)\quad (T - k \log q)^{-1} = \int_0^{\infty} e^{-\rho z (T - k \log q)} z d\rho,$$

which is valid if z is any complex number such that $\Re(z(T-k \log q)) > 0$.

We reduce the size of δ if necessary, and choose numbers $\gamma_{p+1}, \gamma_{p+2}, \cdots$ and D having the property stated in Lemma 4d.

Let $z_m = e^{i\gamma_m}$ $(m = p+1, p+2, \cdots)$. Then if T is on $\mathcal{J}_m(\delta)$,

$$(64)\quad \begin{aligned} \Re(z_m(T - k \log q)) &= \Re(z_m T) - k \log q \Re(z_m) \geqq \Re(z_m T) \\ &= \Re(T) \cos \gamma_m - \Im(T) \sin \gamma_m > M_1 + Dm. \end{aligned}$$

In particular, $\Re(z_m(T-k \log q)) > 0$ for every non-negative k.

Let T be on $\mathcal{J}_m(\delta)$. Then, letting S be an abbreviation for $T-k \log q$, we have

$$(65)\quad S^{-1} = \int_0^{\infty} e^{-\rho z_m S} z_m d\rho.$$

Hence

$$(66)\quad \begin{aligned} Y_m(x, b) &= \frac{1}{2\pi i} \int_{\mathcal{J}_m(\delta)} \frac{F(T)}{T - B} \left[\sum_{k=0}^{\infty} \phi_k (x - b)^k (k \log q - B) \cdot \int_0^{\infty} e^{-\rho z_m S} z_m d\rho \right] dT \\ &= \frac{1}{2\pi i} \int_{\mathcal{J}_m(\delta)} \frac{F(T)}{T - B} \left[\int_0^{\infty} \sum_{k=0}^{\infty} e^{-\rho z_m S} \phi_k (x - b)^k \cdot (k \log q - B) z_m d\rho \right] dT. \end{aligned}$$

(The change of order of limiting operations is justified by the convergence of the integral

$$(67)\qquad \int_0^{\infty}\sum_{k=0}^{\infty}|e^{-\rho z_m S}||\phi_k||x-b|^k|k\log q-B|\,d\rho.$$

See Appendix, Note A.) Hence

$$(68)\qquad Y_m(x,b)=\frac{1}{2\pi i}\int_{\mathcal{J}_m(\delta)}\frac{F(T)}{T-B}\int_0^{\infty}e^{-\rho z_m T}\Phi(\rho)z_m\,d\rho\,dT$$

where

$$\Phi(\rho)=\sum_{k=0}^{\infty}\phi_k(x-b)^k(k\log q-B)e^{\rho z_m k\log q}.$$

Therefore

$$(69)\qquad Y_m(x,b)=\frac{1}{2\pi i}\int_{\mathcal{J}_m(\delta)}\frac{F(T)}{T-B}\int_0^{\infty}e^{-\rho z_m T}(\theta_m(\rho)\phi'(\xi)-B\phi(\xi))z_m\,d\rho\,dT,$$

where

$$(70)\qquad \theta_m(\rho)=e^{\rho z_m\log q}(x-b)\log q,\quad\text{and}\quad \xi=b+\theta_m(\rho)/\log q.$$

The integral with respect to ρ in (69), which from the preceding discussion is convergent provided x lies in a suitable neighborhood of b, can now be seen to converge, if b is sufficiently large, for every x in $\mathcal{S}(b,\beta)$, uniformly in every closed bounded subset of $\mathcal{S}(b,\beta)$, and therefore (69) gives an analytic continuation throughout $\mathcal{S}(b,\beta)$ of the function $Y_m(x,b)$ as defined in (62). (Cf. Appendix, Note B.)

Let T_m be a point on $\mathcal{J}_m(\delta)$ at minimum distance from B. Let W_m be a point on $\mathcal{J}_m(\delta)$ for which $\Re(z_mT)$ is minimum. Let δ_1 be a positive number less than β, such that $\Gamma_1,\cdots,\Gamma_h$ are in the open interval $(-[\beta-\delta_1],\beta-\delta_1)$. (Cf. Lemma 4d.) Then if x is in $\mathcal{S}(b,\beta-\delta_1)$, it follows from Lemma 7e that $\xi\in\mathcal{S}(b,\beta-\delta_1)$. Hence

$$(71)\qquad |Y_m(x,b)|<\frac{(N-1)h(x)\delta}{\lambda(\delta)|T_m-B|}\int_0^{\infty}e^{-\rho E_m}d\rho,$$

where $h(x)=[|x-b||\log q|C_1(\epsilon,\delta_1)+|B|C_0(\epsilon,\delta_1)]e^{(M+\epsilon)|x|}$ and $E_m=\Re(z_mW_m)-(M+\epsilon)q^{-1}$. (The justification of this statement is indicated by the discussion in Appendix, Note B.) Hence

$$|Y_m(x,b)|<\frac{(N-1)\delta h(x)}{\lambda(\delta)|T_m-B|E_m}\cdot$$

Now

$$(72)\qquad \Re(z_mW_m)-M_1>Dm\qquad (m=p+1,p+2,\cdots).$$

This follows from the definition of z_m. Hence if ϵ is sufficiently small and b is sufficiently large, $E_m > Dm$. Also, there is a positive constant D_1 such that

$$|T_m - B| > D_1 m. \tag{73}$$

This follows from (72). Thus

$$|Y_m(x, b)| < \frac{(N-1)\delta h(x)}{\lambda(\delta) D D_1} m^{-2}. \tag{74}$$

Hence $\sum_{m=p+1}^{\infty} Y_m(x, b)$ converges in $\mathfrak{S}(b, \beta)$, uniformly for x in $\mathfrak{S}(b, \beta - \delta_1)$. Since (by Lemma 7d) $\mathfrak{S}(b, \beta - \delta_1)$ includes a neighborhood of b, $\sum_{m=p+1}^{\infty} Y_m(x, b)$ converges in $\mathfrak{S}(b, \beta)$ to a function which is an analytic continuation of $\sum_{m=p+1}^{\infty} Y_m(x, b)$ as defined using (62).

We now consider the function $G_s(x, b)$ defined in (61). Let x, x_0 be points of $\mathfrak{S}_0(\beta)$, such that the line segment joining x to x_0 does not pass through b. Let $u = x - b$ and let $u_0 = x_0 - b$. Then, if x and x_0 are inside the circle of convergence of the series for $\phi(x)$ in powers of $x - b$, we have[13]

$$G_s(x, b) = G_s^*(x, b) + H(x, b), \tag{75}$$

where

$$G^*(x, b) = \int_{u_0}^{u} \frac{1}{2\pi i} \int_{\mathcal{J}_s(\delta)} \Phi(t, T, x)\, dT\, dt \tag{76}$$

$(s = 1, 2, \cdots, p)$, where

$$\Phi(t, T, x) = (u/t)^{T/\log q} \left(\frac{\phi(t+b)}{t \log q} \right) F(T).$$

This follows from a straightforward computation of $G_s^*(x, b)$ by the method of residues. (Cf. Appendix, Note C.)

Also, if x_1 and x_2 are any two complex numbers (independent of x), lying in $\mathfrak{S}_0(\beta)$, and such that the line segment joining them does not pass through b, then

$$\int_{u_1}^{u_2} \frac{1}{2\pi i} \int_{\mathcal{J}_s(\delta)} \Phi(t, T, x)\, dT\, dt = H(x, b). \tag{77}$$

(Cf. Appendix, Note D.) If x_2 is taken as zero, while x_1 is taken as x_0, then by addition of (75) and (77) we obtain

$$G_s(x, b) = G_s^{**}(x, b) + H(x, b), \tag{78}$$

where

$$G_s^{**}(x, b) = \int_{-b}^{u} \frac{1}{2\pi i} \int_{\mathcal{J}_s(\delta)} \Phi(t, T, x)\, dT\, dt, \tag{79}$$

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[13] See the footnote on equation (56).

or

$$G_s^{**}(x, b) = \int_0^x \frac{1}{2\pi i}\int_{\mathcal{J}_s(\delta)} \frac{e^{TV}\phi(\xi)}{(\xi - b)\log q} F(T)dTd\xi \tag{80}$$

(where $V = [\log(x-b) - \log(\xi - b)]/(\log q)$), for all x sufficiently near b. Moreover, the right-hand member of (80) is defined and analytic throughout [14] $\mathfrak{S}(\beta)$, and therefore (78) gives an analytic continuation throughout[14] $\mathfrak{S}(\beta)$ of $G_s(x, b)$ as defined by (61).

Let

$$y_0(x, b) = g\phi(x) + \sum_{s=1}^{p} G_s^{**}(x, b) - \sum_{m=p+1}^{\infty} Y_m(x, b), \tag{81}$$

where g is defined by (60), and $Y_m(x, b)$ is defined by (69). Then $y_0(x, b) + H(x, b)$ is an analytic continuation throughout[15] $\mathfrak{S}(b, \beta)$ of $y(x, b)$ as defined in (52).

It is readily verified (cf. Appendix, Note E) that

(82) As b becomes infinite, $y_0(x, b)$ approaches, uniformly in every closed bounded subset of $\mathfrak{S}(\beta)$, the function $y_0(x)$ as defined by

$$y_0(x) = g\phi(x) + \sum_{s=1}^{p} G_s^{**}(x) - \sum_{m=p+1}^{\infty} Y_m(x), \tag{83}$$

where

$$G_s^{**}(x) = \int_0^x \frac{1}{2\pi i}\int_{\mathcal{J}_s(\delta)} e^{(x-\xi)T}\phi(\xi)F(T)dTd\xi \tag{84}$$

($s = 1, 2, \cdots, p$), and

$$Y_m(x) = \frac{1}{2\pi i}\int_{\mathcal{J}_m(\delta)} \frac{F(T)}{T - B}\int_0^\infty e^{-\rho z_m T}\phi_m(x, \rho)z_m d\rho dT, \tag{85}$$

where $\phi_m(x, \rho) = \phi'(x + z_m\rho) - B\phi(x + z_m\rho)$.

Now $\sum A_j y_0(T_j x, b)$, which equals $\phi(x)$, is near $\sum A_j y_0(x + \omega_j)$, uniformly on every closed bounded subset of $\mathfrak{S}(\beta)$, if b is large, since $T_j x$ is near $x + \omega_j$ (by Lemma 9c), and $y_0(x + \omega_j, b)$ is near $y_0(x + \omega_j)$. Hence $y_0(x)$ is a solution of (1). Evidently $y_0(x)$ is analytic in $\mathfrak{S}(\beta)$. It is easy to see that (83), (84), (85) define a function $y_0(x)$ which is analytic at the origin as well as in $\mathfrak{S}(\beta)$. Hence $y_0(x)$ is analytic in $\mathfrak{S}_0(\beta)$.

The next step is to secure an estimate of $G_s^{**}(x)$. Evidently

$$G_s^{**}(x) = \sum_{j=1}^{j_s} \frac{B_{sj}}{(j-1)!}\int_0^x e^{(x-\xi)\zeta_s}(x - \xi)^{j-1}\phi(\xi)d\xi. \tag{86}$$

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(14) More accurately, throughout $\mathfrak{S}(\beta)$ deprived of the half-line $\{x; x \geqq b\}$.

(15) More accurately, throughout $\mathfrak{S}(b, \beta)$ cut along the half-line $\{x; x \geqq b\}$.

If $|x|=R$ and if $|\xi|=r$, and if $\arg x=\gamma$, then

$$G_s^{**}(x) = \sum_{j=1}^{j_s} \frac{B_{sj}}{(j-1)!}\int_0^R e_s(r)\phi(re^{i\gamma})dr \tag{87}$$

where

$$e_s(r) = e^{(R-r)\zeta_s \exp(i\gamma)}(R-r)^{j-1}e^{ij\gamma},$$

and therefore, if $\delta_1<\beta-|\gamma|$, we have

$$|G_s^{**}(x)| \leqq \sum_{j=1}^{j_s} \frac{|B_{sj}|}{(j-1)!}\int_0^R E_s(r)C_0(\epsilon,\delta_1)e^{(M+\epsilon)r}dr \tag{88}$$

where $E_s(r) = e^{(R-r)\Re[\zeta_s \exp(i\gamma)]}(R-r)^{j-1}$. Since ζ_s is in $\mathcal{N}(M,\beta)$, we have

$\Re[\zeta_s \exp(i\gamma)]\leqq M$. Hence

$$\begin{aligned} |G_s^{**}(x)| &\leqq \sum_{j=1}^{j_s} \frac{|B_{sj}|}{(j-1)!}\int_0^R e^{(R-r)M}(R-r)^{j-1}C_0(\epsilon,\delta_1)e^{(M+\epsilon)r}dr \\ &\leqq \sum_{j=1}^{j_s} \frac{|B_{sj}|}{(j-1)!} e^{(M+\epsilon)R}R^jC_0(\epsilon,\delta_1) \end{aligned} \tag{89}$$

or

$$|G_s^{**}(x)| \leqq \sum_{j=1}^{j_s} \frac{|B_{sj}|}{(j-1)!} e^{(M+\epsilon)|x|}|x|^jC_0(\epsilon,\delta_1). \tag{90}$$

The next step is to secure an estimate of $Y_m(x)$. Let δ_1 be sufficiently small so that z_m is in $\mathfrak{S}(\beta-\delta_1)$ $(m=p+1, p+2, \cdots)$. (We recall that there are only finitely many distinct numbers z_m.) Then if x is in $\mathfrak{S}(\beta-\delta_1)$, $x+z_m\rho$ is in $\mathfrak{S}(\beta-\delta_1)$, and consequently for every positive ϵ

$$|\phi(x+z_m\rho)| < C_0(\epsilon,\delta_1)e^{(M+\epsilon)(|x|+\rho)},$$

and $|\phi'(x+z_m\rho)|<C_1(\epsilon,\delta_1)e^{(M+\epsilon)(|x|+\rho)}$ (where $C_1(\epsilon,\delta_1)$ is the symbol introduced in equation (71)). Thus

$$|Y_m(x)| < \frac{(N-1)\delta}{\lambda(\delta)}\frac{1}{D_1m}\int_0^\infty e^{-\rho(M_1+Dm)}K(x,\rho)d\rho \tag{91}$$

where

$$K(x,\rho) = [C_1(\epsilon,\delta_1)+|B|C_0(\epsilon,\delta_1)]e^{(M+\epsilon)(|x|+\rho)},$$

and therefore, if ϵ is sufficiently small so that $M+\epsilon<M_1$, we have

$$|Y_m(x)| < \frac{(N-1)\delta e^{(M+\epsilon)|x|}}{\lambda(\delta)DD_1m^2}[C_1(\epsilon,\delta_1)+|B|C_0(\epsilon,\delta_1)]. \tag{92}$$

Hence

$$(93) \qquad \left| \sum_{m=p+1}^{\infty} Y_m(x) \right| < \frac{(N-1)\delta}{\lambda(\delta)} \frac{C_1(\epsilon, \delta_1) + |B| C_2(\epsilon, \delta_1)}{D_1 D} \cdot e^{(M+\epsilon)|x|} \left(\sum_{m=p+1}^{\infty} m^{-2} \right).$$

It follows from (90) and (93) that $y_0(x)$ is of type (M, β).

Let $H(x)$ be generic notation for a function of the form

$$(94) \qquad H(x) = \sum_{s=1}^{p} \sum_{j=0}^{j_s-1} C_{sj} x^j e^{\zeta_s x}.$$

It is easily verified that $H(x)$ is always of type (M, β), and is always a solution of the equation $\sum A_j y(x+\omega_j) = 0$. Hence every function of the form

$$(95) \qquad y_0(x) + H(x)$$

is a solution of (1) of type (M, β).

It remains to prove the converse, namely that every solution of (1) of type (M, β) is of the form (95).

Let $Y(x)$ be any solution of (1) of type (M, β). Let

$$\eta(x) = Y(x) - y_0(x).$$

We shall show that $\eta(x) = H(x)$.

Evidently $\eta(x)$ is of type (M, β). Let $d_0(\epsilon, \delta_1)$, $d_1(\epsilon, \delta_1)$, $d_2(\epsilon, \delta_1)$ be such that $|\eta(x)| < d_0(\epsilon, \delta_1) e^{|x|(M+\epsilon)}$ in $\mathfrak{S}(\beta - \delta_1)$, $|\eta'(x)| < d_1(\epsilon, \delta_1) e^{|x|(M+\epsilon)}$ in $\mathfrak{S}(\beta - \delta_1)$, and $|\eta''(x)| < d_2(\epsilon, \delta_1) e^{|x|(M+\epsilon)}$ in $\mathfrak{S}(\beta - \delta_1)$. (We are using Lemma 12.)

Obviously $\eta(x)$ is a solution of the equation

$$(96) \qquad \sum_{j=1}^{n} A_j \eta(x + \omega_j) = 0.$$

Hence $\eta(x)$ is a solution of the equation

$$(97) \qquad \sum_{j=1}^{n} A_j \eta(T_j x) = \psi(x, b),$$

where the T_j are defined by (50), and where

$$(98) \qquad \psi(x, b) = \sum_{j=1}^{n} A_j [\eta(T_j x) - \eta(x + \omega_j)].$$

Moreover $\eta(x)$ is analytic in $\mathfrak{S}(\beta)$. Hence $\psi(x, b)$ is analytic at $x = b$. Thus, if

$$(99) \qquad \eta(x) = \sum_{k=0}^{\infty} \eta_k (x - b)^k$$

and

$$\psi(x, b) = \sum_{k=0}^{\infty} \Psi_k(x-b)^k, \tag{100}$$

then (by comparison of like powers of $x-b$ in (97)), we have

$$\eta_k f(k \log q) = \Psi_k. \tag{101}$$

Hence

$$\begin{aligned} \eta_k &= \Psi_k F(k \log q) \quad \text{if} \quad f(k \log q) \neq 0, \quad \text{and} \\ \Psi_k &= 0 \quad \text{if} \quad f(k \log q) = 0. \end{aligned} \tag{102}$$

Consequently

$$\eta(x) = \sum_{\Psi_k \neq 0} \Psi_k F(k \log q)(x-b)^k + H(x, b). \tag{103}$$

Also,

(104) There is a positive β_0 less than β, such that $\zeta_1, \cdots, \zeta_p$ are the zeros of $f(z)$ in $\mathcal{N}(M, \beta_0)$, such that the numbers $\omega_2, \cdots, \omega_n, z_{p+1}, z_{p+2}, \cdots$ are in $S(\beta_0)$, and such that $\psi(x, b)$ is of type (M, β_0). Moreover, if β_1 is any number such that $\beta_0 < \beta_1 < \beta$, then there is a positive number b_1 such that $T_j x \in S(\beta_1)$ $(j=1, 2, \cdots, n)$, when $x \in S(\beta_0)$ and $b > b_1$. (Cf. Appendix, Note F.)

Using (58) we write

$$\eta(x) = g\psi(x, b) + \sum_{s=1}^{p} E_s(x, b) - \sum_{m=p+1}^{\infty} R_m(x, b) + H(x, b), \tag{105}$$

where g is defined by (60), and

$$E_s(x, b) = \sum_{\Psi_k \neq 0} \Psi_k P_s(k \log q)(x-b)^k \tag{106}$$

$(s=1, 2, \cdots, p)$, and

$$R_m(x, b) = \frac{1}{2\pi i} \int_{\mathcal{J}_{m(\delta)}} \frac{F(T)}{T-B} \Omega(x, b, T) dT \tag{107}$$

with

$$\Omega(x, b, T) = \sum_{k=0}^{\infty} \left(\frac{k \log q - B}{T - k \log q} \right) \Psi_k (x-b)^k.$$

Then in $S(b, \beta_0)$ we have, as in the treatment of $Y_m(x, b)$,

$$R_m(x, b) = \frac{1}{2\pi i} \int_{\mathcal{J}_{m(\delta)}} \frac{F(T)}{T-B} \int_0^{\infty} e^{-\rho z_m T} W(\rho, x, b) z_m d\rho dT \tag{108}$$

where

$$W(\rho, x, b) = e^{\rho z m \log q}(x - b) \log q\psi'(\zeta, b) - B\psi(\zeta, b)$$

with $\zeta = b + e^{\rho z m \log q}(x-b)$ and with $\psi'(\zeta, b)$ the derivative of $\psi(\zeta, b)$ with respect to ζ. And in $\mathfrak{S}(\beta_0)$ we have, as in the treatment of $G_s(x)$,

$$(109) \qquad E_s(x, b) = E_s^{**}(x, b) + H(x, b)$$

where

$$(110) \qquad E_s^{**}(x, b) = \int_0^x \frac{1}{2\pi i}\int_{\mathcal{J}_s(\delta)} e^{VT}\left(\frac{\psi(\xi, b)F(T)}{(\xi - b)\log q}\right) dTd\xi,$$

with $V = [\log(x-b) - \log(\xi - b)]/(\log q)$. (It will be noted that (106) is an exact analogue of (61), since, by (97), $\Psi_k = 0$ whenever $k = \sigma_s$.) Now

$$(111) \qquad |\psi(x, b)| \leqq \sum_{j=1}^{n} |A_j| \, |T_j x - (x + \omega_j)| \, G_j,$$

if $x \in \mathfrak{S}(\beta_0)$, where

(112) G_j is the maximum of $|\eta'(t)|$ for t on the line segment $(T_j x, x + \omega_j)$. But $|T_j x| \leqq |x| + q^{-1}|\omega_j|$, and $|x + \omega_j| \leqq |x| + |\omega_j|$. Hence $|t| \leqq |x| + q^{-1}|\omega_j|$. Also, if $x \in \mathfrak{S}(\beta_0)$, $x + \omega_j \in \mathfrak{S}(\beta_0)$ and $T_j x \in \mathfrak{S}(\beta_1)$. Hence $t \in \mathfrak{S}(\beta_1)$. Therefore if $\delta_1 = \beta - \beta_1$, we have

$$(113) \qquad |\eta'(t)| \leqq d_1(\epsilon, \delta_1)e^{(M+\epsilon)|t|}.$$

Hence

$$(114) \qquad G_j \leqq d_1(\epsilon, \delta_1)e^{(M+\epsilon)q^{-1}|\omega_j|}e^{(M+\epsilon)|x|}.$$

Now by Lemma 9c, with $z = \omega_j$, we have

$$(115) \qquad |T_j x - (x + \omega_j)| \leqq |\omega_j| \, (|x| + |\omega_j| + 1)b^{-1}q^{-2}.$$

Hence, by (111), (114) and (115) we have, when $x \in \mathfrak{S}(\beta_0)$, and $b > b_1 + 1$,

$$(116) \qquad |\psi(x, b)| \leqq b^{-1}d^*(\epsilon)e^{(M+2\epsilon)|x|},$$

where $d^*(\epsilon)$ is independent of x and b.

Let β_2 be a positive number less than β_0 such that $\zeta_1, \cdots, \zeta_p$ are the zeros of $f(z)$ in $\mathfrak{N}(M, \beta_2)$ and such that the numbers $\omega_2, \cdots, \omega_n, z_{p+1}, z_{p+2}, \cdots$, are in $\mathfrak{S}(\beta_2)$. Then by the Cauchy integral formula, and (116), we have, using the analyticity of ψ at $x = 0$, $|\psi'(x, b) \leqq b^{-1}d^{**}(\epsilon)e^{(M+2\epsilon)|x|}$ in $\mathfrak{S}(\beta_2)$, where $d^{**}(\epsilon)$ is independent of x and b. Hence

$$(117) \qquad |R_m(x, b)| \leqq b^{-1}\frac{(N-1)\delta}{\lambda(\delta)D_1 D}[(1 + b^{-1}|x|)d^{**}(\epsilon) + |B| d^*(\epsilon)] \cdot \left[\frac{e^{(M+2\epsilon)|x|}}{m^2}\right],$$

whence

$$\left|\sum_{m=p+1}^{\infty} R_m(x, b)\right| \leqq b^{-1}J(x), \tag{118}$$

where $J(x)$ is a positive-valued function of x which is independent of b and which is bounded on every bounded subset of $\mathfrak{S}(\beta_2)$.

The next step is to estimate $E_s^{**}(x, b)$. We have

$$|e^{VT}| \leqq e^{|T||V|}. \tag{119}$$

But $|V| \leqq I(x, b) = b|x|(b-2|x|)^{-1}$ by Lemma 10a and $|T| \leqq |\zeta_s| + \delta \leqq G_0$, if $T \in \mathcal{J}_s(\delta)$ $(s=1, 2, \cdots, p)$, with G_0 a positive number. Therefore

$$|E_s^{**}(x)| \leqq \frac{e^{I(x,b)G_0}d^*(\epsilon)e^{(M+2\epsilon)|x|}b^{-1}|x|\delta}{\lambda(\delta)(1 - b^{-1}|x|)} \leqq b^{-1}I_s(x) \tag{120}$$

where $I_s(x)$ is a positive-valued function of x which is independent of b and which is bounded on every fixed bounded subset of $\mathfrak{S}(\beta_2)$ if b is sufficiently large. Let

$$\eta_0(x, b) = g\psi(x, b) + \sum_{s=1}^{p} E_s^{**}(x, b) - \sum_{m=p+1}^{\infty} R_m(x, b). \tag{121}$$

By virtue of inequalities (118), (120), and (116) it is evident that as b becomes infinite $\eta_0(x, b)$ tends uniformly in every bounded subset of $\mathfrak{S}(\beta_2)$ to a limit function which is identically zero.

Now from (105), (109) and (121) we have

$$\eta(x) = \eta_0(x, b) + H(x, b), \tag{122}$$

and since $\eta(x)$ is independent of b, it follows that $H(x, b)$ tends to the limit $\eta(x)$ as b becomes infinite, the limit being approached uniformly in every bounded subset of $\mathfrak{S}(\beta_2)$. But this implies that $\eta(x)$ is of the form $H(x)$. (See Lemma 11.) This concludes the uniqueness proof.

Let $\mathcal{L}_m$ $(m=p+1, p+2, \cdots)$ be the contour $z=\rho z_m$ $(0 \leqq \rho)$ described in the sense of increasing ρ. Then equations (42)–(45) are valid.

This completes the proof of Theorem 2.

Remarks. (i) The methods used in the proof of Theorem 2 can be thought of as a generalization of the method of principal solutions ("in direction 0") which was introduced by the author [14]. The generalization consists in that the exponents $\zeta_s/\log q$ appearing in $H(x, b)$ are not required to be real and non-negative. There are other deviations of the methods of this paper from the methods of the earlier paper: among them the use of powers of the functions $\log (1-b^{-1}x)/\log q$ in the solutions of (51), and the failure to introduce parameters in (51). These other deviations, however, are matters of convenience only, and could have been avoided.

(ii) From (45) it is evident that $Y_m(x)$ can be written in the form

$$\int_{\mathcal{L}_m} [\phi'(x+z) - B\phi(x+z)]H_m(z)dz,$$

where

$$H_m(z) = (2\pi i)^{-1}\int_{\mathcal{J}_{m(\delta)}} F(T)(T-B)^{-1}e^{-zT}dT.$$

If we let $Z_1, \cdots, Z_Q$ be the distinct numbers in the set $\{z_{p+1}, z_{p+2}, \cdots\}$, and define $I_t(z)$ as $\sum_{z_m=Z_t} H_m(z)$ (this series will converge for z on $\mathcal{L}_m$, if $z\neq 0$), $t=1, 2, \cdots, Q$, and define $\mathcal{C}_t$ as the contour $\mathcal{L}_m$ for $z_m=Z_t$, and define $K_s(z)$ as $(2\pi i)^{-1}\int_{\mathcal{J}_s(\delta)} e^{zT}F(T)dT$, then we shall have

$$y_0(x) = g\phi(x) + \sum_{s=1}^{p}\int_0^x K_s(x-\xi)\phi(\xi)d\xi$$
$$+ \sum_{t=1}^{Q}\int_{\mathcal{C}_t} [\phi'(x+z) - B\phi(x+z)]I_t(z)dz.$$

In this form the representation of $y_0(x)$ bears a close analogy to Nörlund's representation of his principal solution of equation (4), [8, p. 70, equation (9)].

(iii) It is evident from the proof of Theorem 2 that the *existence* problem for equation (1) might have been approached from the standpoint of the symbolic calculus, in this fashion: One writes (1) in the form $f(D)y=\phi$, whence, formally, $y=F(D)\phi$, and then, again formally, from Theorem 1a, $F(D)=F(B)-\sum_{m=1}^{\infty}(2\pi i)^{-1}\int_{\mathcal{J}_{m(\delta)}} F(T)(T-B)^{-1}(D-B)(T-D)^{-1}dT$, leading to

$$y = F(B)\phi(x) - \sum_{m=1}^{\infty} Y_m^*(x),$$

where

$$Y_m^*(x) = (2\pi i)^{-1}\int_{\mathcal{J}_{m(\delta)}} F(T)(T-B)^{-1}h(x, T)dT,$$

and where $h(x, T)=(D-B)(T-D)^{-1}\phi(x)$, that is, where $h(x, T)$ is a solution of the differential equation $Th(x, T)-h'(x, T)=\phi'(x)-B\phi(x)$. Evidently $Y_m(x)$, as defined by (85), is a particular case of $Y_m^*(x)$ $(m=p+1, p+2, \cdots)$.

The approximating q-difference method appearing in this paper provides analytic calculations with q, paralleling these formal calculations with the symbolic operator D. From this point of view the approximating q-difference method plays a role analogous to that played by the Laplace transform in the treatment of differential or difference equations(16).

(16) Cf., for example, Doetsch [4, chap. 18, §3].

Part VI. Appendix.

Note A (referring to the integral (67)).

We observe that $|\exp(-\rho z_m(T-k\log q))| \leqq \exp(-\rho\Re(z_m T)) \leqq \exp(-\rho M_1)$, while the power series $\sum_{k=0}^{\infty}|\phi_k|\,|x-b|^k|k\log q-B|$, which is independent of ρ, is convergent when $|x-b|<b\sin\beta$ (by virtue of the analyticity of $\phi(x)$ in $\mathfrak{S}(\beta)$).

Note B (referring to the paragraph following equation (70)).

We observe that since x is in $\mathfrak{S}(b, \beta)$, and since $-\rho z_m \log q$ is in $\mathfrak{S}(\beta)$, it follows from Lemma 7e that ξ is in $\mathfrak{S}(b, \beta)$, and thus (by Lemma 7b) ξ is in $\mathfrak{S}(\beta)$. Hence $\phi(\xi)$ and $\phi'(\xi)$ are defined, and (by Lemma 12), for suitably small positive δ_1 corresponding to any fixed closed bounded subset of $\mathfrak{S}(b, \beta)$, $|\phi(\xi)|<C_0(\epsilon, \delta_1)\exp[(M+\epsilon)|\xi|]$, and $|\phi'(\xi)|<C_1(\epsilon, \delta_1)\exp[(M+\epsilon)|\xi|]$. Then, since $|\xi|<|x|+q^{-1}\rho$ (by Lemma 9b), we have $|\phi(\xi)|<C_0(\epsilon, \delta_1)\exp[(M+\epsilon)(|x|+q^{-1}\rho)]$ and $|\phi'(\xi)|<C_1(\epsilon, \delta_1)\exp[(M+\epsilon)(|x|+q^{-1}\rho)]$. Since $|\exp(-\rho z_m T)\leqq\exp(-\rho M_1)$, it suffices that ϵ be sufficiently small and b sufficiently large so that $(M+\epsilon)q^{-1}<M_1$ in order to insure convergence of the integral with respect to ρ, in (69), for every x in $\mathfrak{S}(b, \beta)$, uniformly in every closed bounded subset of $\mathfrak{S}(b, \beta)$.

Note C (referring to equations (75) and (76) and the following sentence).

Evidently

$$G_s^*(x, b) = \int_{u_0}^{u} 2(\pi i)^{-1}\int_{\mathcal{J}_s(\delta)} \Gamma(t, T, x)P_s(T)dTdt,$$

where

$$\Gamma(t, T, x) = e^{WT}\phi(t+b)(t\log q)^{-1},$$

with $W=(\log u-\log t)\log q$. Hence

$$G_s^*(x, b) = \int_{u_0}^{u} \phi(t+b)(t\log q)^{-1}R(x, t)dt,$$

where $R(x, t)$ is the residue at ζ_s of $e^{WT}P_s(T)$.

Now

$$R(x, t) = e^{W\zeta_s}\cdot\frac{1}{2\pi i}\int_{\mathcal{J}_s(\delta)}\left[\left(\sum_{m=0}^{\infty}\frac{(T-\zeta_s)^m W^m}{m!}\right)\left(\sum_{j=1}^{j_s}B_{sj}(T-\zeta_s)^{-j}\right)\right]dT$$

$$= e^{W\zeta_s}\sum_{j=1}^{j_s}\frac{B_{sj}W^{j-1}}{(j-1)!}.$$

Hence

$$G_s^*(x, b) = \int_0^{W_0}\sum_{k=0}^{\infty}\phi_k u^k e^{W(\zeta_s - k\log q)}\sum_{j=1}^{j_s}\frac{B_{sj}W^{j-1}}{(j-1)!}dW$$

where $W_0 = (\log u - \log u_0)/(\log q)$. Hence

$$G_s^*(x, b) = \sum_{k \neq \sigma_s} \phi_k u^k \sum_{j=1}^{j_s} \frac{B_{sj}}{(j-1)!} [I(s, j, k) + (j-1)!(k \log q - \zeta_s)^{-j}] + \psi_s u^{\sigma_s} \sum_{j=1}^{j_s} (B_{sj} W_0^j)/(j!),$$

where

$$I(s, j, k) = - e^{W_0(\zeta_s - k \log q)} \sum_{m=0}^{j-1} \frac{W_0^{j-1-m}(j-1)!}{(k \log q - \zeta_s)^{m+1}(j-1-m)!}.$$

Thus

$$\begin{aligned} G_s^*(x, b) &= \sum_{k \notin (\sigma_1, \cdots, \sigma_p)} \phi_k u^k P_s(k \log q) + \psi_s u^{\sigma_s} \frac{B_{s,j_s}}{j_s!} \left(\frac{\log (1 - b^{-1}x)}{\log q} \right)^{j_s} \\ &\quad + H(x, b) \\ &= G_s(x, b) + H(x, b). \end{aligned}$$

Note D (referring to equation (77)).

By the argument of Note C, the left-hand member of (77) equals

$$\int_{u_1}^{u_2} \phi(t + b)(t \log q)^{-1}(u/t)^{\sigma_s} \sum_{j=1}^{j_s} B_{sj}(\log u - \log t)^{j-1} L_j dt,$$

where $L_j = ((j-1)!)^{-1}(\log q)^{1-j}$. Hence the left-hand member of (77) equals

$$u^{\sigma_s} \sum_{m=0}^{j_s - 1} (\log u)^m C_m = H(x, b)$$

where

$$C_m = \int_{u_1}^{u_2} \phi(t + b) t^{-(1+\sigma_s)} \sum_{j=1}^{j_s} D_{sj}(-\log t)^{j-1-m} dt$$

with $D_{sj} = B_{sj}(\log q)^{-j}((j-1-m)!)^{-1}(m!)^{-1}$.

Note E (referring to statement (82)).

Let $\mathfrak{F}$ be any closed bounded subset of $\mathfrak{S}(\beta)$. Let P be the maximum of $|x|$ in $\mathfrak{F}$, and let b be greater than $3P$. Let δ_1 be positive and sufficiently small so that $\mathfrak{F} \subset \mathfrak{S}(\beta - \delta_1)$. Let x be any point of $\mathfrak{F}$.

We note that

$$G_s^{**}(x, b) - G_s^{**}(x) = \int_0^x (2\pi i)^{-1} \int_{\mathcal{J}_s(\delta)} J(x, \xi, T, b) \phi(\xi) F(T) dT d\xi \tag{E.1}$$

where

$$J(x, \xi, T, b) = (\xi - b)^{-1}(\log q)^{-1} e^{VT} - e^{(x-\xi)T},$$

with V having the same significance as it had in Lemma 10.

Evidently

$$J(x, \xi, T, b) = (\xi - b)^{-1} (\log q)^{-1} (e^{VT} - e^{(x-\xi)T}) + ((\xi - b)^{-1}(\log q)^{-1} - 1)e^{(x-\xi)T}.$$

But

$$e^{VT} - e^{(x-\xi)T} = T \int_{x-\xi}^{V} e^{\alpha T} d\alpha.$$

Hence $|e^{VT} - e^{(x-\xi)T}| \leqq |T| \, |V - (x-\xi)| \exp (R|T|)$ where $R = \max (|V|, |x-\xi|)$.

By Lemma 10a, $|V| < Pb(b-2P)^{-1}$. By Lemma 10b, $|V-(x-\xi)| < 2(P+1)^2 q^{-1}(b-2P)^{-1}$.

Hence $|e^{VT} - e^{(x-\xi)T}| \leqq b^{-1}K(\mathfrak{F})$ where $K(\mathfrak{F})$ is a positive number depending upon $\mathfrak{F}$ but independent of b and T, the inequality holding for all T on $\mathcal{J}_s(\delta)$ $(s=1, 2, \cdots, p)$.

Also, $|(\xi-b)^{-1}(\log q)^{-1}| \leqq b(b-P)^{-1}$ (by Lemma 8), and $|(\xi-b)^{-1}(\log q)^{-1} - 1| = |(\xi-b)^{-1}(\log q)^{-1}| \, |(1+b \log q) - \xi \log q| \leqq b(b-P)^{-1}[(2qb)^{-1} + P(qb)^{-1}]$ (by Lemma 8). Thus $|J(x, \xi, T, b)| \leqq K_1(\mathfrak{F})b^{-1}$, where $K_1(\mathfrak{F})$ is a positive number depending upon $\mathfrak{F}$ but independent of b and T, the inequality holding for all T on $\mathcal{J}_s(\delta)$ $(s=1, 2, \cdots, p)$.

Since $\phi(\xi)$ and $F(T)$, in (E.1), are bounded by a bound independent of b, we conclude that $G_s^{**}(x, b)$ approaches $G_s^{**}(x)$ as b becomes infinite, uniformly in $\mathfrak{F}$.

Now

$$Y_m(x, b) - Y_m(x) = (2\pi i)^{-1} \int_{\mathcal{J}_m(\delta)} F(T)(T - B)^{-1} \int_0^{\infty} \exp(-\rho z_m T) d_m(x, \rho, T, b) z_m d\rho dT$$

where

$$d_m(x, \rho, T, b) = U_m\phi'(\xi_m) - B\phi(\xi_m) - [\phi'(x + z_m\rho) - B\phi(x + z_m\rho)],$$

with $\xi_m = b + (x-b) \exp(\rho z_m \log q)$ and with $U_m = (x-b) \log q \exp(\rho z_m \log q)$. Evidently

$$d_m(x, \rho, T, b) = (U_m - 1)\phi'(\xi_m) + [\phi'(\xi_m) - \phi'(x + z_m\rho)] - B[\phi(\xi_m) - \phi(x + z_m\rho)].$$

Let δ_1 be sufficiently small so that every z_m is in $\mathfrak{S}(\beta - \delta_1)$. (We recall that there are only finitely many distinct z_m.) Let b be sufficiently large so that $\mathfrak{F}$ is in $\mathfrak{S}(b, \beta - \delta_1)$. (Cf. Lemma 7c.) Then since x is in $\mathfrak{S}(b, \beta - \delta_1)$, so is ξ_m (by Lemma

7e), and therefore both $x+z_m\rho$ and ξ_m are in $S(\beta-\delta_1)$. (Cf. Lemma 7b.) Now $U_m-1=x\log q\exp(\rho z_m\log q)-(b\log q+1)-b\log q(\exp(\rho z_m\log q)-1)$. Hence by Lemmas 8 and 9a we have $|U_m-1|\leqq b^{-1}q^{-1}|x|+(2qb)^{-1}+(bq^2)^{-1}\rho \leqq(bq^2)^{-1}(|x|+\rho+1)$. Also $|\phi'(\xi_m)|\leqq C_1(\epsilon,\delta_1)\exp[(M+\epsilon)|\xi_m|]$, and since (by Lemma 9b) $|\xi_m|\leqq|x|+q^{-1}\rho$, we have $|\phi'(\xi_m)|\leqq C_1(\epsilon,\delta_1)\exp[(M+\epsilon)q^{-1}\cdot(|x|+\rho)]$. Now $\phi'(\xi_m)-\phi'(x+z_m\rho)=\int_{x+z_m\rho}^{\xi_m}\phi''(t)dt$, and since t, on the line segment $(x+z_m\rho,\ \xi_m)$, is not more than $|x|+q^{-1}\rho$, we conclude that $|\phi'(\xi_m)-\phi'(x+z_m\rho)|\leqq|\xi_m-(x+z_m\rho)|C_2(\epsilon,\delta_1)\exp[(M+\epsilon)q^{-1}(|x|+\rho)]$. By Lemma 9c we have $|\xi_m-(x+z_m\rho)|\leqq\rho(|x|+\rho+1)(bq^2)^{-1}$. Hence $|\phi'(\xi_m)-\phi'(x+z_m\rho)|\leqq K_2(\mathfrak{F})(\rho+1)^2b^{-1}\exp[\rho(M+\epsilon)q^{-1}]$, where $K_2(\mathfrak{F})$ is a positive number independent of b and ρ, the inequality holding for every m and for every nonnegative ρ. Similarly $|\phi(\xi_m)-\phi(x+z_m\rho)|\leqq K_3(\mathfrak{F})(\rho+1)^2b^{-1}\exp[\rho(M+\epsilon)q^{-1}]$, where $K_3(\mathfrak{F})$ has properties similar to those of $K_2(\mathfrak{F})$. Thus $|d_m(x,\rho,T,b)|\leqq K_4(\mathfrak{F})(\rho+1)^2b^{-1}\exp[\rho(M+\epsilon)q^{-1}]$, where $K_4(\mathfrak{F})$ has properties similar to those of $K_2(\mathfrak{F})$. From this it follows that if ϵ is sufficiently small and b is sufficiently large, $|d_m(x,\rho,T,b)|\leqq b^{-1}K_5(\mathfrak{F})\exp(\rho M_1)$. Hence, if T_m is a point of $\mathcal{J}_m(\delta)$ minimizing $|T-B|$, and W_m is a point of $\mathcal{J}_m(\delta)$ minimizing $\Re(z_mT)$, then $|Y_m(x,b)-Y_m(x)|\leqq(N-1)\delta[\lambda(\delta)]^{-1}|T_m-B|\cdot b^{-1}K_5(\mathfrak{F})\int_0^\infty[e^{-\rho\Re(z_mW_m)}\cdot e^{\rho M_1}]d\rho\leqq b^{-1}K_6(\mathfrak{F})m^{-2}$ (using (72) and (73) in obtaining the last inequality). Thus

$$\left|\sum_{m=p+1}^{\infty}Y_m(x,b)-\sum_{m=p+1}^{\infty}Y_m(x)\right|\leqq b^{-1}K_6(\mathfrak{F})\sum_{m=p+1}^{\infty}m^{-2},$$

which shows that as b becomes infinite $\sum_{m=p+1}^{\infty}(Y_m(x),b)$ approaches $\sum_{m=p+1}^{\infty}Y_m(x)$, uniformly in $\mathfrak{F}$. This, together with what has been proved about the convergence of $G_s^{**}(x,b)$ to $G_s^{**}(x)$, proves that as b becomes infinite $y_0(x,b)$ approaches $y_0(x)$ uniformly in every closed bounded subset of $S(\beta)$.

Note F (referring to statement (104)).

Let β_0 be a positive number less than β, such that $\zeta_1,\cdots,\zeta_p$ are the zeros at $f(z)$ in $\mathcal{N}(M_1,\beta_0)$. (Cf. Lemma 4e.) Let β_0 be sufficiently large so that $\omega_2,\cdots,\omega_n$ and $z_{p+1},z_{p+2},\cdots$ are in $S(\beta_0)$. (We recall that there are only finitely many distinct numbers z_m.) Let β_1 be such that $\beta_0<\beta_1<\beta$.

Let x be in $S(\beta_0)$. We have $T_jx=b(1-q^{\omega_j})+q^{\omega_j}x$. Now $b(1-q^{\omega_j})$ is in $S(b,\beta_0)$ (by definition of $S(b,\beta_0)$, and therefore $b(1-q^{\omega_j})$ is in $S_0(\beta)$ by Lemma 7b. Also, $\arg(q^{\omega_j}x)=\arg x+\Im[\omega_j\log q]$. Let b_0 be a positive number such that if $b>b_0$, then $|\Im[\omega_j\log q]|<\beta_1-\beta_0$ $(j=1,2,\cdots,n)$. Then if $b>b_0$ and x is in $S(\beta_0)$, $q^{\omega_j}x$ is in $S(\beta_1)$. Hence T_jx is in $S(\beta_1)$ $(j=1,2,\cdots,n)$.

Also, $|T_jx|\leqq|x|+b|1-q^{\omega_j}|\leqq|x|+|\omega_j|/q$ (by Lemma 9a).

Let ϵ be any positive number. Let $\delta_1=\beta-\beta_1$. If $b>b_0$ and if x is in $S(\beta_0)$, then $|\eta(T_jx)|<d_0(\epsilon,\delta_1)\exp[|T_jx|(M+\epsilon)]\leqq d_0(\epsilon,\delta_1)\exp(|x|(M+\epsilon))\exp((M+\epsilon)|\omega_j|/q)$.

Obviously $\eta(T_jx)$ is analytic if $x=0$. Hence $\eta(T_jx)$ is of type (M,β_0). Hence $\psi(x,b)$ is of type (M,β_0).

Bibliography

1. S. Bochner, *Hauptlösungen von Differenzengleichungen*, Acta Math. vol. 51 (1928) pp. 1–21.

2. É. Borel, *Contribution a l'étude des fonctions méromorphes*, Ann. École Norm. (3) vol. 18 (1901) pp. 211–239.

3. R. D. Carmichael, *Systems of linear difference equations and expansions in series of exponential functions*, Trans. Amer. Math. Soc. vol. 35 (1933) pp. 1–28.

4. G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, New York, 1943.

5. M. Ghermanesco, *Sur les equations aux difference finies*, Acta Math. vol. 62 (1934) pp. 239–287.

6. G. Halphén, *Sur quelques séries pour le developpement des fonctions à une seule variable*, Bull. Sci. Math. (2) vol. 5 (1881) pp. 462–488.

7. R. E. Langer, *On the zeros of exponential sums and integrals*, Bull. Amer. Math. Soc. vol. 37 (1931) pp. 213–239.

8. N. E. Nörlund, *Vorlesungen über differenzenrechnung*, Berlin, 1924.

9. S. Pincherle, *Sur la resolution de l'équation fonctionnelle* $\sum h_\nu \phi(x+\alpha_\nu)=f(x)$ *à coefficients constants*, Acta Math. vol. 48 (1926) pp. 279–304 (first published in 1888).

10. R. Raclis, *Solution principale de l'équation linéaire aux différences finies*, Acta Math. vol. 55 (1930) pp. 277–394.

11. I. M. Sheffer, *A local solution of the difference equation* $\Delta y(x)=F(x)$ *and of related equations*, Trans. Amer. Math. Soc. vol. 39 (1936) pp. 345–379.

12. ———, *A correction*, Trans. Amer. Math. Soc. vol. 41 (1937) pp. 153–159.

13. W. Strodt, *Analytic solutions of non-linear difference equations*, Ann. of Math. vol. 44 (1943) pp. 375–396.

14. ———, *Principal solutions of difference equations*, Amer. J. Math. vol. 69 (1947) pp. 717–757.

15. J. M. Whittaker, *A theorem on meromorphic functions*, Proc. London Math. Soc. (2) vol. 40 (1935–1936) pp. 255–272.

Columbia University,
New York, N. Y.